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Solutions of the time-dependent Klein–Gordon equation in a Schwarzschild background space

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Abstract. Time-dependent solutions of the Klein–Gordon equation for a massive scalar meson field in a Schwarzschild background space are obtained by the use of asymptotic methods. The solutions are found for all angular momentum states and are valid over all space exterior in a Schwarzschild radius provided the black hole is large, and the energy is very much less than the rest-mass energy of the scalar π meson.

1. Introduction

In a recent paper (Rowan and Stephenson 1976, to be referred to as I), the static solutions of the Klein–Gordon equation in a Schwarzschild background space were obtained by asymptotic methods. These methods are now applied to the solution of the time-dependent Klein–Gordon equation in the same space. The solutions, which involve Whittaker functions, are valid for the energy $E \ll m_\pi c^2$, where m_π is the rest mass of the scalar π meson, and are required in the analysis of quantum field theory in a Schwarzschild space. Unlike previous work (see, for example, Boulware 1975) in which solutions have been obtained near and very far from the event horizon, the solutions given here are valid over all space exterior to the Schwarzschild radius provided this is large compared to the Compton wavelength of the π meson.

2. Basic equations

We start with the source-free Klein–Gordon equation

$$(\Box^2+\mu^2)\Phi=0, \tag{2.1}$$

where, as usual, μ is the inverse Compton wavelength of the scalar meson associated with the field. In generally covariant form (2.1) is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\,g^{ik}\frac{\partial\Phi}{\partial x^k}\right)+\mu^2\Phi=0, \tag{2.2}$$

which, with a Schwarzschild background metric

$$ds^2=[1-(2m/r)]\,dt^2-[1-(2m/r)]^{-1}\,dr^2-r^2\,d\theta^2-r^2\sin^2\theta\,d\phi^2, \tag{2.3}$$

where m is the mass of the gravitating body, leads to the equation

$$\left[\frac{r}{r-2m}\frac{\partial^2}{\partial t^2}-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r(r-2m)\frac{\partial}{\partial r}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\mu^2\right]\Phi=0. \tag{2.4}$$

Separating the variables by writing

$$\Phi=\sum_{l,m}\psi_{l,m}(r,t)Y_{l,m}(\theta,\phi), \tag{2.5}$$

where $Y_{l,m}(\theta,\phi)$ are the spherical harmonic functions, and then letting

$$\psi_{l,m}(r,t)=\int R_{l,m,E}(r)\,\mathrm{e}^{\mathrm{i}Et}\,\mathrm{d}E, \tag{2.6}$$

we finally obtain the radial equation for $R_{l,m,E}(r)$ in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r(r-2m)\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\left(l(l+1)-\frac{r^3E^2}{r-2m}+\mu^2r^2\right)R=0. \tag{2.7}$$

Using the substitution $x=(r-2m)/m$ (as in I), (2.7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(x+2)\frac{\mathrm{d}R}{\mathrm{d}x}\right)-\left(l(l+1)+N^2(x+2)^2-\frac{P^2}{x}(x+2)^3\right)R=0, \tag{2.8}$$

where $P=Em$ and $N=\mu m$.

Writing (2.8) in normal form by putting

$$R(x)=[x(x+2)]^{-1/2}Z(x), \tag{2.9}$$

we have

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2}-\left[N^2\left(\frac{2+x}{x}\right)+\frac{l(l+1)}{x(x+2)}-P^2\left(\frac{2+x}{x}\right)^2-\frac{1}{x^2(x+2)^2}\right]Z=0. \tag{2.10}$$

It is this equation which we now wish to solve. There are two ways (at least) of approaching this problem, one of which depends on the reduction to a simpler equation using the asymptotic method due to Liouville and Green as discussed in I. The other method is based on a technique involving an examination of the transition points of (2.10) (see Olver 1974). This method will not be discussed in the present paper.

3. Solution by the Liouville–Green method

In equation (2.10) we first change the independent variable from x to ξ by the transformation $x=x(\xi)$, and then let $G=(\xi')^{1/2}Z$ (where primes denote differentiation with respect to x) to obtain

$$\frac{\mathrm{d}^2G}{\mathrm{d}\xi^2}=\left\{\left[N^2\left(\frac{2+x}{x}\right)+\frac{l(l+1)}{x(x+2)}-P^2\left(\frac{2+x}{x}\right)^2-\frac{1}{x^2(x+2)^2}\right]\frac{1}{\xi'^2}+\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}\right\}G. \tag{3.1}$$

Writing $\alpha^2=l(l+1)/k^2$, $\beta^2=N^2/k^2$, where $k^2=N^2+l(l+1)$ and choosing

$$\xi'^2=\left[\beta^2\left(\frac{2+x}{x}\right)+\frac{\alpha^2}{x(x+2)}\right] \tag{3.2}$$

(3.1) becomes

$$\frac{d^2G}{d\xi^2}=\left[k^2-P^2\left(\frac{2+x}{x}\right)^2\frac{1}{\xi'^2}-\frac{1}{x^2(x+2)^2}\frac{1}{\xi'^2}+\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}\right]G. \tag{3.3}$$

The behaviour of the terms

$$-\frac{1}{x^2(x+2)^2\xi'^2}+\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4} \tag{3.4}$$

has been fully discussed in I, where it is shown that (3.4) may be written as $-1/4\xi^2+g_1(\xi)$, where $g_1(\xi)$ is a slowly varying function and is bounded by at most 5. The remaining term on the right-hand side of (3.3) besides k^2 is

$$-P^2\left(\frac{2+x}{x}\right)^2\frac{1}{\xi'^2}=-P^2\left(\frac{2+x}{x}\right)^2\frac{x(x+2)}{\beta^2(x+2)^2+\alpha^2}. \tag{3.5}$$

For small x, we have from (3.2)

$$\xi\sim\sqrt{2x}(4\beta^2+\alpha^2)^{1/2} \tag{3.6}$$

so that the term (3.5) behaves like $-16P^2/\xi^2$. Equation (3.3) may now be written as

$$\frac{d^2G}{d\xi^2}=\left(k^2-\frac{1}{4\xi^2}-\frac{16P^2}{\xi^2}+g_1(\xi)+g_2(\xi)\right), \tag{3.7}$$

where

$$g_2(\xi)=\frac{16P^2}{\xi^2}-\frac{P^2}{x}(2+x)^3\frac{1}{\beta^2(x+2)^2+\alpha^2}. \tag{3.8}$$

Now

$$g_2(\xi)\sim\begin{cases}-P^2/\beta^2 & \text{as } \xi\to\infty,\\ -\dfrac{64P^2(1+\beta^2)}{3(4\beta^2+\alpha^2)} & \text{as } \xi\to 0.\end{cases} \tag{3.9}$$

By numerical calculation, it is found that $|g_2(\xi)|<16P^2/\beta^2$ and that $g_2(\xi)$ is a slowly varying function. Since we want to neglect $g_2(\xi)$, it must be small compared with k^2. This requirement leads to the condition

$$P\ll N \qquad \text{or} \qquad E\ll\mu, \tag{3.10}$$

which in conventional units is $E\ll m_\pi c^2$, where m_π is the rest mass of the scalar π meson. Neglecting both $g_1(\xi)$ and $g_2(\xi)$ in (3.7) we have finally

$$\frac{d^2G}{d\xi^2}=\left(k^2-\frac{\frac{1}{4}+16P^2}{\xi^2}\right)G. \tag{3.11}$$

Writing $u=2k\xi$, (3.11) becomes

$$\frac{d^2G}{du^2}=\left(\frac{1}{4}-\frac{\frac{1}{4}+16P^2}{u^2}\right)G, \tag{3.12}$$

which is to be compared with the Whittaker equation

$$\frac{d^2G}{du^2}=\left(\frac{1}{4}-\frac{\kappa}{u}+\frac{m^2-\frac{1}{4}}{u^2}\right)G, \tag{3.13}$$

whose solution is

$$G(u)=AM_{\kappa,m}(u)+BM_{\kappa,-m}(u), \tag{3.14}$$

where A and B are arbitrary constants, and (provided $2m$ is not a negative integer)

$$M_{\kappa,m}(u)=u^{m+\frac{1}{2}}\,e^{-u/2}\left(1+\frac{\frac{1}{2}+m-\kappa}{1!(2m+1)}u+\frac{(\frac{1}{2}+m-\kappa)(\frac{3}{2}+m-\kappa)}{2!(2m+1)(2m+2)}u^2+\ldots\right). \tag{3.15}$$

The solution of (3.12) is therefore

$$G(u)=AM_{0,4iP}(u)+BM_{0,-4iP}(u). \tag{3.16}$$

A convenient representation for $M_{\kappa,m}$ when $\kappa=0$ is the Kummer series (see Whittaker and Watson 1927)

$$M_{0,m}(u)=u^{m+\frac{1}{2}}\left(1+\sum_{p=1}^{\infty}\frac{u^{2p}}{2^{4p}p!(m+1)\ldots(m+p)}\right). \tag{3.17}$$

Furthermore it is known that the $M_{0,m}$ functions may be expressed in terms of Bessel functions and that

$$M_{0,im}(u)=\Gamma(1+im)2^{2im}u^{1/2}I_{im}(u/2), \tag{3.18}$$

where I_{im} is the modified Bessel function of the first kind of order im, and Γ is the gamma function. The appearance of Bessel functions of imaginary order in similar work on quantum field theory in curved space has been noted by other authors (see, for example, Boulware 1975).

The final solutions for the radial function $R(x)$ are therefore

$$R(x)=\frac{1}{[x(x+2)]^{1/2}}\frac{[x(x+2)]^{1/4}}{[\beta^2(x+2)^2+\alpha^2]^{1/4}}\begin{cases}M_{0,4iP}(2k\xi),\\ M_{0,-4iP}(2k\xi),\end{cases} \tag{3.19}$$

where, from (3.2),

$$\xi=\int_0^x\left[\beta^2\left(\frac{2+x}{x}\right)+\frac{\alpha^2}{x(x+2)}\right]^{1/2}dx. \tag{3.20}$$

Again we emphasize that this solution has been obtained on the assumption that $E\ll\mu$ (see (3.10)) and N is a large parameter as in I; however, it is valid for all l values.

The calculation of the error due to neglecting the $g(\xi)$ in (3.7) for all ξ in $0\leq\xi<\infty$ is difficult, and it is probably better to examine each case separately rather than to try to obtain a general error formula. It seems very unlikely that the error here, or in I, could be large since $|g_1(\xi)|+|g_2(\xi)|\ll k$.

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